TWO-DIMENSIONAL STEADY CURRENTS IN A NONLINEAR CONDUCTING MEDIUM

I. M. Rutkevich

We consider the simplest two-dimensional problem of the current distribution in a nonlinear conducting medium over a plane wall consisting of a semiinfinite electrode and an insulator.

In calculating steady electric fields in some media it is necessary to take account of the dependence of the electrical conductivity on current. A two-temperature plasma with nonequilibrium ionization is an example of such a medium [1, 2].

When the conductivity is a given function of coordinates, the electric potential can in principle be determined by solving a linear equation of the elliptic type [3]. When the conductivity depends on the current density j, the equations of electrodynamics become nonlinear [4, 5], which considerably complicates the problem. When the Hall effect can be neglected, these equations can be linearized [5] by a transformation in the plane of the hodograph of the vector j.

When the law of conductivity is nearly linear, for example, at sufficiently small current densities, linearized equations can be used [8].

The hodograph method is used to obtain an exact solution for an arbitrary nonlinear law of variation of conductivity. Formally the problem is analogous to the gasdynamic problem of the flow past a plate. A particular law of variation of conductivity is analyzed.

1. We consider the motion of an isotropic conducting liquid with a given velocity field $\mathbf{V} = (u(y), 0, 0)$ at right angles to a constant magnetic field $\mathbf{H}_0 = (0, 0, H_0)$, $H_0 = \text{const}$. The basic equations of stationary electrodynamics are

$$\operatorname{rot} \mathbf{E} = 0, \qquad \operatorname{div} \mathbf{j} = 0 \tag{1.1}$$

If the magnetic Reynolds number R_m is small, the induced magnetic field can be neglected in Ohm's law, so that

$$\mathbf{j} = \sigma(j)(\mathbf{E} + c^{-1}\mathbf{V}\times\mathbf{H}_0) \equiv \sigma(j)\,\mathbf{q} \tag{1.2}$$

Since Eq. (1.2) determines a relation between j and q, the conductivity σ can be regarded as a function of q, which we assume henceforth. Since $\operatorname{rot}(\mathbf{V}\times\mathbf{H}_0) = 0$ for the vectors $\mathbf{V}$ and $\mathbf{H}_0$ as defined above, $\mathbf{q}$ will be a potential vector: $\mathbf{q} = -\nabla\varphi$. Assuming that the current flow is confined to the xy plane, it is easy to obtain from (1.1) a quasilinear equation of the second order for φ:

$$\left[1 + \frac{\delta}{q^2}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\delta}{q^2}\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\,\partial y} + \left[1 + \frac{\delta}{q^2}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\frac{\partial^2\varphi}{\partial y^2} = 0 \qquad \left(\delta = \frac{d\ln\sigma}{d\ln q}\right) \tag{1.3}$$

The last equation can also be rewritten in the form

$$\left(1 + \frac{\delta q_x^2}{q^2}\right)\frac{\partial^2\varphi}{\partial x^2} + 2\delta\frac{q_x q_y}{q^2}\frac{\partial^2\varphi}{\partial x\,\partial y} + \left(1 + \frac{\delta q_y^2}{q^2}\right)\frac{\partial^2\varphi}{\partial y^2} = 0$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 49-53, November-December, 1970. Original article submitted June 2, 1970.

© 1971 Consultants Bureau, a division of Plenum Publishing Corporation, 227 West 17th Street, New York, N. Y. 10011. All rights reserved. This article cannot be reproduced for any purpose whatsoever without permission of the publisher. A copy of this article is available from the publisher for $15.00.

In this form the equation will also be satisfied by the flow function $\psi(x, y)$ whose existence is ensured by the second equation of (1.1), so that $j_x = \partial\psi/\partial y$, $j_y = -\partial\psi/\partial x$.

The type of Eq. (1.3) is determined by the sign of the expression $\Delta = 1 + \delta(q)$. The equation will be elliptic for $\Delta > 0$ and hyperbolic for $\Delta < 0$. The current distribution corresponding to the hyperbolic region is unstable [7]. The condition that (1.3) be elliptic is given in [5, 7] in terms of the relation $\sigma = \sigma_1(j)$ in the form $\delta_1 = d\ln\sigma/d\ln j < 1$. It is easy to show that the last inequality is equivalent to the condition $\delta > -1$. We note also that the region in which (1.3) is elliptic corresponds to the rising part of the j(q) curve, and the hyperbolic region to the falling part.

We turn from the physical xy plane to the plane of the hodograph of the vector **q**, taking as new independent variables q and θ defined by

$$q = \sqrt{q_x^2 + q_y^2}, \quad \operatorname{tg}\theta = \frac{q_y}{q_x}$$

In the plane of the hodograph the flow function ψ satisfies the linear equation

$$\frac{\partial^2\psi}{\partial q^2} + \frac{1+\delta(q)}{q^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1-\delta(q)}{q}\frac{\partial\psi}{\partial q} = 0 \tag{1.4}$$

Equation (1.4) is somewhat simpler than the one satisfied by the "potential" $\varphi(q, \theta)$. If $\psi(q, \theta)$ is a known solution of Eq. (1.4), the transition to the physical plane is accomplished by integrating the simple linear equations

$$\frac{\partial x}{\partial q} = \frac{d}{dq}\left(\frac{1}{\sigma q}\right)\frac{\partial\psi}{\partial\theta}\cos\theta - \frac{1}{\sigma q}\frac{\partial\psi}{\partial q}\sin\theta, \qquad \frac{\partial x}{\partial\theta} = \frac{1}{\sigma}\frac{\partial\psi}{\partial q}\cos\theta - \frac{1}{\sigma q}\frac{\partial\psi}{\partial\theta}\sin\theta$$

$$\frac{\partial y}{\partial q} = \frac{d}{dq}\left(\frac{1}{\sigma q}\right)\frac{\partial\psi}{\partial\theta}\sin\theta + \frac{1}{\sigma q}\frac{\partial\psi}{\partial q}\cos\theta, \qquad \frac{\partial y}{\partial\theta} = \frac{1}{\sigma}\frac{\partial\psi}{\partial q}\sin\theta + \frac{1}{\sigma q}\frac{\partial\psi}{\partial\theta}\cos\theta \tag{1.5}$$

After the functions $x(q, \theta)$ and $y(q, \theta)$ have been found from (1.5), they are used to find the inverse functions $q(x, y)$ and $\theta(x, y)$. Performing the inverse transformation is generally difficult, but the knowledge of the functions $x(q, \theta)$ and $y(q, \theta)$ permits one to extract the basic information on the character of the solution.

2. An analogy between the equations of stationary electrodynamics in a nonlinear medium and the equations of potential flow of a compressible gas for a given dependence of density on velocity was noted in [5]. There are two ways of comparing an electric current with the flow of a gas. The first way corresponds to the relation $\sigma(j)$ and the transition to the plane of the hodograph of the vector **j**. The electric-current lines correspond to the equipotential lines in gas flow. The second way uses the relation $\sigma(q)$ and the transition to the plane of the hodograph of the vector **q**. In this case the flow lines of the gas correspond to the electric-current lines, conductivity to the density of the gas, and $-\delta$ to the square of the local Mach number.

This analogy permits the use of certain gasdynamic solutions to describe the current distribution in a nonlinear conducting medium. In this case the main physical interest is in the electrodynamic analogies of subsonic flows.

Direct verification shows that for an arbitrary $\sigma(q)$ the function

$$\psi(q, \theta) = kq^{-1}\sin\theta, \quad k = \text{const} > 0 \tag{2.1}$$

will be a solution of (1.4). The gasdynamic solution corresponding to a flow function of the type (2.1) is discussed by Ringleb [6]. The integration of Eq. (1.5) for a ψ of the form (2.1) leads to

$$x = r(q)\cos 2\theta + a(q), \quad y = r(q)\sin 2\theta$$

$$r(q) = \frac{k}{2\sigma(q)q^2}, \quad a(q) = r(q) - k\int_q^\infty \frac{dq'}{\sigma(q')q'^3} \tag{2.2}$$

Additive constants are omitted in Eqs. (2.2). Study shows that when the ellipticity condition $1 + \delta(q) > 0$ is satisfied, the integral in the expression for $a(q)$ in (2.2) converges at the upper limit and diverges as $q \to 0$. Henceforth we assume that the ellipticity condition is satisfied.

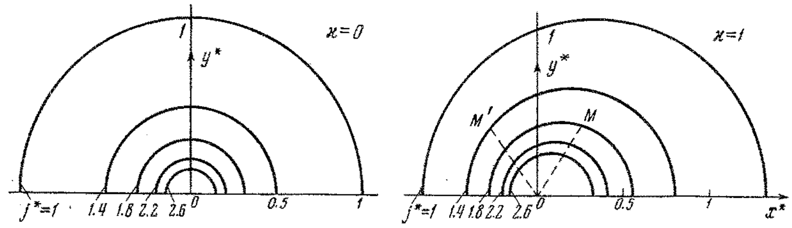

Fig. 1

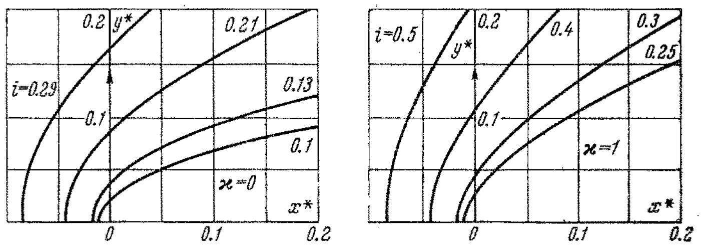

Fig. 2

For the solution (2.2) the Jacobian of the transformation is $D(x, y)/D(q, \theta) = -k\sigma^{-1}q^{-3}(\Delta\cos^2\theta + \sin^2\theta)$. The quantities q and θ are single-valued functions of x and y at all points in the physical plane except the origin.

When $\theta = \pi/2$ and q varies from 0 to ∞, the points in the physical plane range from left to right along the ray $x < 0$, $y=0$. When $\theta = 0$ and q varies from 0 to ∞, the points range from right to left along the ray $x > 0$, $y=0$. Therefore when $y \geq 0$, Eqs. (2.2) determine the current distribution in the half-plane bounded by a semiinfinite electrode and an insulator. The constant k appearing in the solution can be determined if, for example, the potential difference between the electrode $x < 0$, $y=0$ and some point outside this electrode is specified.

The level lines of the absolute magnitude of the effective electric field **q** and of the quantities j and σ form a family of circles of radii r(q) with centers at the points $(a(q), 0)$. It is easy to see that $|a| < r$, and therefore for any q the circles contain the origin. As $q \to \infty$, the quantities $a(q)$ and r(q) approach zero, so that rather large values of the current density are concentrated in a small region covering the end of the electrode.

As an example let us consider the solution (2.2) corresponding to the following relations:

$$\sigma(q) = \sigma_0 = \text{const} \quad (q < q_1); \quad \sigma(q) = \sigma_0 (q / q_1)^{\varkappa}, \quad \varkappa > -1 \quad (q \geqslant q_1) \tag{2.3}$$

The corresponding relation between σ and j has the form

$$\sigma(j) = \sigma_0 \quad (j < j_1 = \sigma_0 q_1), \qquad \sigma(j) = \sigma_0 \left(\frac{j}{j_1}\right)^{\varkappa/(1+\varkappa)} \quad (j \geqslant j_1)$$

For $\varkappa > 0$ the last relation is in qualitative agreement with data presented in [2] for a nonequilibrium plasma. Using Eqs. (2.3), we obtain for q as a function of the polar coordinates ρ and α of points in the physical plane $x = \rho\cos\alpha$, $y = \rho\sin\alpha$

$$q(\rho, \alpha) = q_1 \left[\frac{\rho}{r_1^2 - a_1^2}\left(\sqrt{r_1^2 - a_1^2\sin^2\alpha} - a_1\cos\alpha\right)\right]^{-1/(2+\varkappa)} \quad \begin{pmatrix} 0 < \rho < \rho_*(\alpha) \\ 0 \leqslant \alpha \leqslant \pi \end{pmatrix}$$

$$q(\rho, \alpha) = q_1 r_1^{1/2}(\rho^2 - 2a_1\rho\cos\alpha + a_1^2)^{-1/4} \quad (\rho \geqslant \rho_*(\alpha),\ 0 \leqslant \alpha \leqslant \pi) \tag{2.4}$$

$$r_1 = r(q_1), \quad a_1 = a(q_1) = r_1\varkappa / (2 + \varkappa), \quad \rho_*(\alpha) = \sqrt{r_1^2 - a_1^2\sin^2\alpha} + a.$$

The function $\rho_*(\alpha)$ in (2.4) determines the polar equation of a circle on which $q = q_1$. The equations determining $\theta(\rho, \alpha)$ have the form

$$\theta(\rho, \alpha) = {}^1/_2\ [\alpha + \arcsin (f \sin \alpha)]$$
$$f(\rho, \alpha) = \varkappa / (2 + \varkappa) \quad (0 < \rho < \rho_*(\alpha),\ 0 \leqslant \alpha \leqslant \pi) \tag{2.5}$$
$$f(\rho, \alpha) = [\varkappa / (2 + \varkappa)]\, r_1 (\rho^2 - 2a_1\rho \cos\alpha + a_1^2)^{-1/2} \quad (\rho \geqslant \rho_*(\alpha),\ 0 \leqslant \alpha \leqslant \pi)$$

Analysis of (2.4) and (2.5) shows the following. In contrast to the case of constant electrical conductivity, q will be a function not only of ρ but also of α, with $q(\rho, \alpha) \neq q(\rho, \pi - \alpha)$; i.e., the q(x, y) distribution becomes asymmetric about x. This is the case also for an arbitrary nonlinear j(q) relation corresponding to the region of ellipticity.

In using Eqs. (2.3) the character of the asymmetries of the q and j distributions are such that the values of these quantities at the symmetrical points M = (x, y) and M' = (−x, y), where x > 0, satisfy the inequalities

$$q(M) > q(M'),\ j(M) > j(M') \qquad (\varkappa > 0)$$
$$q(M) < q(M'),\ j(M) < j(M') \qquad (-1 < \varkappa < 0)$$

If $0 < x < \min(r_1 - a_1, r_1 + a_1), y = 0$, it follows from (2.4) that

$$q(M) / q(M') = (1 + \varkappa)^{1/(2+\varkappa)}, \quad j(M) / j(M') = (1 + \varkappa)^{(1+\varkappa)/(2+\varkappa)}$$
$$g(M)/g(M') = 1 + \varkappa$$

Here g = jq is the local Joule dissipation.

Thus in the practically interesting case when $\varkappa > 0$ the values of the effective electric field, the current density, and the local dissipation at point M near the insulator are larger than the values of the corresponding quantities at the symmetrical point M' near the electrode. Figure 1 shows the level lines of the dimensionless current density $j^* = j/j_1$ in the plane of the dimensionless coordinates $x^* = x/r_1$, $y^* = y/r_1$ for $\varkappa = 0$ (constant conductivity) and $\varkappa = 1$.

As $\rho \to 0$, q and j increase as ρ to the power $-1/(2+\varkappa)$ and $-(1+\varkappa)/(2+\varkappa)$ respectively.* The local dissipation increases the same way as in a linear medium: $g \sim \rho^{-1}$.

It is clear from (2.5) that for $\varkappa > 0$ the flow lines intersect an arbitrary ray α = const at larger angles θ than the corresponding flow lines in a linear medium. Consequently there is less tendency than in the linear case for the flow lines to be turned along the nonconducting wall. On the other hand, when $-1 < \varkappa < 0$, this tendency is increased.

Figure 2 shows the flow lines in the x^*y^* plane for $\varkappa = 0$ and $\varkappa = 1$. The parameter characterizing the family of lines is the magnitude of the dimensionless current i flowing in the tube formed by the flow line and the nonconducting wall.

As a result of using a "stable" $\sigma(q)$ relation in (2.2) there is no region analogous to the region of supersonic flow in the Ringleb solution [6]. Our solution can be used as a qualitative estimate of the effect of various laws of nonlinear conductivity on the end effect close to the junction of the electrode and insulator.

LITERATURE CITED

1. J. L. Kerrebrock, "Conduction in gases with elevated electron temperatures," in: Engineering Aspects of Magnetohydrodynamics, Columbia Univ. Press, New York (1962).
2. J. L. Kerrebrock, "Magnetohydrodynamic generators with nonequilibrium ionization," AIAA Journal, 3, No. 4 (1965).
3. A. B. Vatazhin and S. A. Regirer, "An approximate calculation of the current distribution when a conducting fluid flows along a channel within a magnetic field," Prikl. Matem. i. Mekhan., 26, No. 3 (1962).
4. D. A. Oliver and M. Mitchner, "Nonuniform electrical conduction in MHD channels," AIAA Journal, 5, No. 8 (1967).
5. Yu. P. Emets, "The hodograph method in the electrodynamics of continuous nonlinear conducting media," Prikl. Matem. i. Mekhan., 31, No. 6 (1967).

*The manner in which the current density increases near a singular point when $\sigma(j)$ is a power law was examined also by V. A. Buchin (thesis, MGU, 1969).

6. F. Ringleb, Exakte Lösungen der Differentialgleichungen einer adiabatischen Gasströmung, Z. ang. Math. und Mech., 20, No. 4 (1940).
7. A. G. Kulikovskii and S. A. Regirer, "On the stability and evolution of the electric current distribution in a medium with a nonlinear conductivity," Prikl. Matem. i. Mekhan., 32, No. 4 (1968).
8. V. A. Buchin, "Determination of the Joule dissipation during the motion of a fluid with a variable conductivity in a nonuniform magnetic field," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 6 (1968).